doi:10.1006/jmaa.2001.7624, available online at http://www.idealibrary.com on IDEAL®

KKM and Nash Equilibria Type Theorems in Topological Ordered Spaces

Qun Luo

Department of Mathematics, Zhaoqing University, Zhaoqing, Guangdong 526061, China

Submitted by William A. Kirk

Received March 3, 2000

In this paper, we obatin a generalized KKM theorem, a generalized Fan–Browder fixed theorem, and an existence theorem of Nash equilibria in topological ordered spaces. © 2001 Elsevier Science

1. INTRODUCTION

The classical KKM theorem has numerous applications in various fields of pure and applied mathematics. In 1972, Ky Fan [1] obtained a minimax inequality which plays a fundamental role in nonlinear analysis and mathematical economics (see [3, 4, 6], etc.).

In this paper, by Theorem 2 in [2] (C. D. Horvath and J. V. Llinares Ciscar), we obtain a generalized KKM theorem, a Ky Fan inequality, a Fan–Browder fixed-point theorem, and an existence theorem of Nash equilibria in topological ordered spaces.

2. PRELIMINARIES

A semilattice is a partially ordered set X, with the partial ordering denoted by $\leq$, for which any pair (x, x') of elements has a least upper bound, denoted by $x \vee x'$. It is easy to see that any nonempty finite subset A of X has a least upper bound, denoted by $\sup A$. In a partially ordered set $(X, \leq)$, two arbitrary elements x and x' do not have to be comparable, but, in the case where $x \leq x'$, the set $[x, x'] = \{y \in X : x \leq y \leq x'\}$

0022-247X/01 $35.00

is called an order interval. Now assume that $(X, \leq)$ is a semilattice and $A \subseteq X$ is a nonempty finite subset; then the set $\Delta(A) = \bigcup_{a \in A}[a, \sup A]$ is well defined, and it has the following properties:

(a) $A \subseteq \Delta(A)$,

(b) if $A \subseteq A'$, then $\Delta(A) \subseteq \Delta(A')$.

We shall say that a subset $E \subseteq X$ is Δ-convex if, for any nonempty finite subset $A \subseteq E$, we have $\Delta(A) \subseteq E$.

EXAMPLE. Let $X = \{(x, 1) : 0 \leq x < 1\} \cup \{(x, y) : 0 \leq y \leq 1, x \geq 1, y \geq x - 1\} \subset R^2$. The partial ordering of R^2 is defined by

$$(a, b), (c, d) \in R^2, \qquad (a, b) \leq (c, d) \Longleftrightarrow c - a \geq 0,$$
$$d - b \geq 0, \qquad \text{and} \qquad d - b \leq c - a.$$

Then X is Δ-convex.

For any $D \subset X$, $\mathcal{F}(D)$ denotes the family of all finite subsets of D, $\Delta(D) = \bigcup_{A \in \mathcal{F}(D)} \Delta(A)$.

Let X and Y be two topological spaces. For a binary relation $R \subseteq X \times Y$, we set for $x \in X$ and $y \in Y$,

$$R(x) = \{y \in Y : (x, y) \in R\} \qquad \text{and} \qquad R^{-1}(y) = \{x \in X, (x, y) \in R\}.$$

The following theorem is due to [2].

THEOREM 2.1. *Let X be a topological semilattice with path-connected intervals, let $X_0 \subseteq X$ be a nonempty subset of X, and let $R \subseteq X_0 \times X$ be a binary relation such that*

(i) *For each $x \in X_0$, the set $R(x) = \{y \in X : (x, y) \in R\}$ is not empty and closed in $R(X_0)$.*

(ii) *There exists $x_0 \in X_0$ such that the set $R(x_0)$ is compact.*

(iii) *For any nonempty finite subset $A \subseteq X_0$,*

$$\bigcup_{x \in A} [x, \sup A] \subseteq \bigcup_{x \in A} R(x).$$

Then the set $\bigcap_{x \in X_0} R(x)$ is not empty.

Let X be a nonempty set and let Y be a topological space. 2^Y denotes the family of all subsets of Y. A mapping G: $X \to 2^Y$ is said to be *transfer closed valued* (e.g., see Zhou and Tian [7]) if for each $x \in X$ and $y \notin G(x)$, there exists $x' \in X$ and an open neighborhood $N(y)$ of y in Y such that $y' \notin G(x')$ for each $y' \in N(y)$. It is obvious that if a mapping G: $X \to 2^Y$ is transfer closed valued, then for each $x \in X$ and $y \notin G(x)$, there exists some $x' \in X$ such that $y \notin \mathrm{cl}G(x')$, where $\mathrm{cl}G(x)$ is the closure of $G(x)$. Then G: $X \to 2^Y$ is transfer closed if $\bigcap_{x \in X} G(x) = \bigcap_{x \in X} \mathrm{cl}G(x)$.

Let X, Y be two topological spaces. T: $X \to 2^Y$ is said to have the local intersection property (see [5]) if for each $x \in X$ with $T(x) \neq \emptyset$, there exists an open neighborhood $N(x)$ of x such that $\bigcap_{z \in N(x)} T(z) \neq \emptyset$. It is not hard to derive that T: $X \to 2^Y$ has the local intersection property if and only if $X \backslash T^{-1}$ is transfer closed valued, i.e., $\bigcup_{y \in Y} T^{-1}y = \bigcup_{y \in Y} \operatorname{int}(T^{-1}y)$.

By Theorem 2.1 and the definition of transfer closed, it is easy to obtain the following generalized KKM theorem.

THEOREM 2.2. *Let X be a topological semilattice with path-connected intervals, let $X_0 \subseteq X$ be a nonempty subset of X, and let $R \subseteq X_0 \times X$ be a binary relation such that*

(1) *G: $X_0 \to 2^X$ is transfer closed valued, where $G(x) = \{y \in X : (x, y) \in R\}$ for each $x \in X_0$.*

(2) *There exists $x_0 \in X_0$ such that the set $\mathrm{cl}G(x_0)$ is compact.*

(3) *For any nonempty finite subset $A \subseteq X_0, \bigcup_{x \in A}[x, \sup A] \subseteq \bigcup_{x \in A} G(x)$.*

Then the set $\bigcap_{x \in X_0} G(x)$ is not empty.

Proof. Since $\mathrm{cl}G(x)$ satisfies all of the conditions of Theorem 2.1, then $\bigcap_{x \in X_0} \mathrm{cl}G(x)$ is not empty. Since $\bigcap_{x \in X_0} G(x) = \bigcap_{x \in X_0} \mathrm{cl}G(x)$, hence $\bigcap_{x \in X_0} G(x)$ is not empty. Theorem 2.2 is proved.

Let X be a topological semilattice or a Δ-convex subset of a topological semilattice. f: $X \to (-\infty, +\infty)$ is Δ-quasiconcave if, for any nonempty finite subset $A = \{x_1, x_2, \ldots, x_n\} \subseteq X$, for any $y \in \Delta(A), f(y) \geq \min\{f(x_1), f(x_2), \ldots, f(x_n)\}$. It is easy to see that f: $X \to (-\infty, +\infty)$ is Δ-quasiconcave if and only if the set $\{y \in X : f(y) > \lambda\}$ or the set $\{y \in X : f(y) \geq \lambda\}$ is Δ-convex for any $\lambda \in (-\infty, +\infty)$.

EXAMPLE. Let $X = \{(x, 1) : 0 \leq x \leq 1\} \cup \{(1, y) : 0 \leq y \leq 1\} \subset R^2$. The partial ordering of R^2 is defined by

$$(a, b), (c, d) \in R^2, \qquad (a, b) \leq (c, d) \Longleftrightarrow a \leq c \text{ and } b \leq d.$$

A function f is defined on X by

$$f(z) = f(x, y) = -x^2 - y^2, \qquad \text{for any } z = (x, y) \in X.$$

It is easy to see that f is Δ-quasiconcave.

Let X, Y be two topological spaces. $\varphi(x, y)$: $X \times Y \to (-\infty, +\infty)$ is said to be strongly path transfer lower semicontinuous relative to x (in short, SPT l.s.c) if for each $(x, y) \in X \times Y$ and for all $\varepsilon > 0$, there exists an open neighborhood $N(x)$ of x in X and there exists $y^0 \in Y$ such that for any $x' \in N(x)$,

$$\varphi(x, y) \leq \varphi(x', y^0) + \varepsilon.$$

It is easy to see that any l.s.c. function is SPT l.s.c. The converse is not true.

EXAMPLE. Let $X = [0, 1]$, $Y = [0, 1]$. A function $\varphi(x, y)$ is defined on $X \times Y$ by

$$\varphi(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x = y, \\ 1, & \text{if } y = 0,\ x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\varphi(x, y)$ is not l.s.c on $X \times Y$, and $\varphi(x, y)$ is SPT l.s.c. relative to x.

3. GENERALIZED KY FAN INEQUALITY AND FAN–BROWDER FIXED-POINT THEOREM

We obtain the following generalized Ky Fan inequality by Theorem 2.2.

THEOREM 3.1. *Let X be a topological semilattice with path-connected intervals and let $f: X \times X \to (-\infty, +\infty)$ satisfy*

(1) *For each $x \in X$, $f(x, x) \leq 0$.*

(2) *$f(x, y)$ SPT l.s.c. relative to y.*

(3) *There exists $x_0 \in X$ such that $\mathrm{cl}\{y \in X : f(x_0, y) \leq 0\}$ is a compact set.*

(4) *For each $y \in X$, $x \to f(x, y)$ is Δ-quasiconcave.*

Then there exists $y^* \in X$ such that $f(x, y^*) \leq 0$ for each $x \in X$.

Proof. Let $W = \{(x, y) \in X \times X : f(x, y) \leq 0\}$. $W(x) = \{y \in X : f(x, y) \leq 0\}$ for each $x \in X$, for any $y_0 \notin W(x)$, i.e., $f(x, y_0) > 0$. $\forall\, 0 < \varepsilon < f(x, y_0)$. By (2), there exists a neighborhood $N(y_0)$ of y_0 in X, there exists $x' \in X$ such that $f(x, y_0) < f(x', y') + \varepsilon$ for each $y' \in N(y_0)$, and hence $f(x', y') > 0$, $y' \notin W(x')$; then $W(x)$ is transfer closed valued.

Suppose there exists a nonempty finite subset $A = \{x_1, x_2, \ldots, x_n\} \subseteq X$ such that $\Delta(A) \not\subset \bigcup_{x \in A} W(x)$. There exists $y_0 \in \Delta(A) = \bigcup_{x \in A} [x, \sup A]$ with $y_0 \notin \bigcup_{x \in A} W(x)$, i.e., $y_0 \notin W(x_i), f(x_i, y_0) > 0$ for each $i = 1, 2, \ldots, n$. By (4), $f(y_0, y_0) \geq \min\{f(x_1, y_0), f(x_2, y_0), \ldots, f(x_n, y_0)\} > 0$, which contradicts (1). Then W satisfies all of the conditions of Theorem 2.2, so $\bigcap_{x \in X} W(x) \neq \emptyset$, take $y^* \in \bigcap_{x \in X} W(x)$ and hence $f(x, y^*) \leq 0$ for each $x \in X$. Theorem 3.1 is proved.

Remark 1. Obviously, if X is a compact, then (3) holds.

By Theorem 3.1, we obtain the following generalized Fan–Browder fixed-point theorem.

THEOREM 3.2. *Let X be a topological semilattice with path-connected intervals and let $F\colon X \to X$ be a mapping with nonempty, closed Δ-convex values which has the local intersection property. If there exists $x_0 \in X$ such that* $\mathrm{cl}(X\backslash F^{-1}(x_0))$ *is compact, then F has a fixed point.*[1]

Proof. Let $f\colon X \times X \to (-\infty, +\infty)$,

$$f(x,y) = \begin{cases} 1, & \text{if } x \in F(y), \\ 0, & \text{if } x \notin F(y). \end{cases}$$

Suppose the theorem were not true; then for any $x \in X$, $x \notin F(x)$, hence $f(x,x) = 0$. For any $y \in X$, $\lambda \in (-\infty, +\infty)$,

$$\{x \in X\colon f(x,y) > \lambda\} = \begin{cases} X, & \text{if } \lambda < 0, \\ F(y), & \text{if } 0 \le \lambda < 1, \\ \emptyset, & \text{if } \lambda \ge 1 \end{cases}$$

is Δ-convex valued, and hence $f(\cdot, y)$ is Δ-quasiconcave. Since F has the local intersection property, for $(x,y) \in X \times X$, $F(y) \neq \emptyset$, there exists a neighborhood $N(y)$ of y in X such that $\bigcap_{u \in N(y)} F(u) \neq \emptyset$. For any $\varepsilon > 0$, take $x' \in \bigcap_{u \in N(y)} F(u)$, then $x' \in F(y')$ for any $y' \in N(y)$, so $f(x', y') = 1$, $f(x,y) < 1 + \varepsilon = f(x', y') + \varepsilon$. Thus $f(x,y)$ is SPT l.s.c. relative to y and

$$\mathrm{cl}\{y \in X\colon f(x_0, y) \le 0\} = \mathrm{cl}\{y \in X\colon y \notin F^{-1}(x_0)\} = \mathrm{cl}(X\backslash F^{-1}(x_0))$$

is compact. By Theorem 3.1 there exists $y^* \in X$ such that $f(x, y^*) \le 0$ for each $x \in X$, a contradiction, so there exists $x^* \in X$ such that $x^* \in F(x^*)$. Theorem 3.2 is proved.

Theorem 3.2 implies the following maximal element theorem:

THEOREM 3.3. *Let X be a nonempty compact Δ-convex subset of a topological semilattice with path-connected intervals. Assume that $G\colon X \to 2^X$ has the local intersection property and that for any $x \in X$, $x \notin \Delta(G(x))$. Then there exists $x^* \in X$ such that $G(x^*) = \emptyset$.*

Proof. Since G has the local intersection property, $\Delta(G)\colon X \to 2^X$ also has the local intersection property, where for any $x \in X$, $\Delta(G)(x) = \Delta(G(x)) = \bigcup_{A \in \mathcal{F}(G(x))} \Delta(A)$. Suppose that for any $x \in X$, $G(x) \neq \emptyset$. By Theorem 3.2, there exists $x^0 \in X$ such that $x^0 \in \Delta(G(x^0))$, which is a contradiction. So there exists $x^* \in X$ such that $G(x^*) = \emptyset$. Theorem 3.3 is proved.

[1] $x^* \in X$ is called a fixed point of the mapping F if $x^* \in F(x^*)$.

4. EXISTENCE THEOREM OF NASH EQUILIBRIA

Let $(X_i, \leq_i)$, $i \in I$, be a family of topological semilattices, and let X and $\widehat{X}_i$ be the product space with the product topology, i.e.,

$$X = \prod_{i \in I} X_i, \qquad \widehat{X}_i = \prod_{j \in I \setminus i} X_j,$$

and for x, $x' \in X = \prod_{i \in I} X_i$, defined $x \leq x'$ iff $x_i \leq {}_i x'_i$ for each $i \in I$. $(X, \leq)$ is then a topological semilattice with $(x \bigvee x')_i = x_i \bigvee_i x'_i$ for each $i \in I$. For any $x \in X$, $x = (x_i, \hat{x}_i)$, where $x_i \in X_i$, $\hat{x}_i \in \widehat{X}_i = \prod_{j \in I \setminus i} X_j$.

A point $x^* \in X$ is a Nash equilibrium for a given family of functions f_i: $X \to (-\infty, +\infty)$, if for all $i \in I$,

$$f_i(x_i^*, \hat{x}_i^*) = \max_{u_i \in X_i} f_i(u_i, \hat{x}_i^*).$$

By Theorem 3.2, we obtain the existence theorem of a Nash equilibrium point.

THEOREM 4.1. *Let $N = \{1, 2, \ldots, n\}$, for each $i \in N$, let X_i be a nonempty compact and sequentially compact Δ-convex subset of topological semilattice with path-connected intervals, and $X = \prod_{i \in N} X_i$, $f_i : X \to (-\infty, +\infty)$ satisfying*

(i) *For each $i \in N$, for any $\hat{x}_i \in \widehat{X}_i$, $u_i \to f_i(u_i, \hat{x}_i)$ is Δ-quasiconcave.*

(ii) *For each $i \in N$, f_i is u.s.c.*

(iii) *For each $i \in N$, $f_i(u_i, \hat{x}_i)$ is SPT l.s.c. relative to $\hat{x}_i$.*

Then there exists $x^* \in X$ such that for each $i \in N$

$$f_i(x_i^*, \hat{x}_i^*) = \max_{u_i \in X_i} f_i(u_i, \hat{x}_i^*).$$

Proof. For any $k = 1, 2, 3, \ldots$, let W_k: $X \to 2^X$, $W_k(x) = \prod_{i \in N} T_i(x)$. For any $x \in X$, where T_i: $X \to 2^{X_i}$,

$$T_i(x) = \left\{ y_i \in X_i : f_i(y_i, \hat{x}_i) > \max_{u_i \in X_i} f_i(u_i, \hat{x}_i) - \frac{1}{k} \right\}.$$

So $W_k(x) \neq \emptyset$ and is Δ-convex for any $x \in X$. In the following, we prove that $W_k(x)$ has the local intersection property, i.e., if $W_k(x) \neq \emptyset$, there exists an open neighborhood $O(x)$ of x in X such that $\bigcap_{u \in O(x)} W_k(u) \neq \emptyset$. Since $\bigcap_{u \in O(x)} \prod_{i \in N} T_i(u) = \prod_{i \in N} \bigcap_{u \in O(x)} T_i(u)$, we only need to prove that for each $i \in N$, T_i has the local intersection property. Let $T_i(x^0) \neq \emptyset$; take $y^0 \in T_i(x^0)$, i.e.,

$$f_i(y_i^0, \hat{x}_i^0) > \max_{u_i \in X_i} f_i(u_i, \hat{x}_i^0) - \frac{1}{k}. \tag{1}$$

For any $0 < \varepsilon < (f_i(y_i^0, \hat{x}_i^0) - (\max_{u_i \in X_i} f_i(u_i, \hat{x}_i^0) - \frac{1}{k}))/2$, by (iii), there exists an open neighborhood $O_1(\widehat{x}_i^0)$ of $\hat{x}_i^0$ in $\widehat{X}_i$ and there exists $y_i^* \in X_i$ such that

$$f_i(y_i^0, \hat{x}_i^0) < f_i(y_i^*, \hat{x}_i') + \varepsilon \qquad \forall\, \hat{x}_i' \in O_1(\hat{x}_i^0). \tag{2}$$

And since $\max_{u_i \in X_i} f_i(u_i, \widehat{x}_i)$ is continuous at $\hat{x}_i^0$, there exists an open neighborhood $O_2(\hat{x}_i^0)$ of $\hat{x}_i^0$ in $\widehat{X}_i$ such that for any $\hat{x}_i' \in O_2(\hat{x}_i^0)$

$$\max_{u_i \in X_i} f_i(u_i, \hat{x}_i^0) - \varepsilon < \max_{u_i \in X_i} f_i(u_i, \hat{x}_i') < \max_{u_i \in X_i} f_i(u_i, \hat{x}_i^0) + \varepsilon. \tag{3}$$

Take $O(x^0) = O(x_i^0) \times (O_2(\hat{x}_i^0) \cap O_1(\hat{x}_i^0))$, where $O(x_i^0)$ denotes an open neighborhood of x_i^0 in X_i. By (1), (2), and (3), for any $x' \in O(x^0)$,

$$f_i(y_i^*, \hat{x}_i') > f_i(y_i^0, \hat{x}_i^0) - \varepsilon > \max_{u_i \in X_i} f_i(u_i, \hat{x}_i^0) - \frac{1}{k} + \varepsilon > \max_{u_i \in X_i} f_i(u_i, \hat{x}_i') - \frac{1}{k}.$$

Then $y_i^* \in \bigcap_{u \in O(x^0)} T_i(u) \neq \emptyset$ and hence $\bigcap_{u \in O(x^0)} T_i(u) \neq \emptyset$, so T_i has the local intersection property, and hence so does W_k. By Theorem 3.2, there exists $x^k \in X$ such that $x^k \in W_k(x^k)$, since X is compact, and $\{x^k\}_{k=1}^{\infty}$ has a cluster point $x^* \in X$. Without loss of generality, we may assume that $x^k \to x^*(k \to \infty)$, i.e., for each $i \in N$,

$$f_i(x_i^k, \hat{x}_i^k) > \max_{u_i \in X_i} f_i(u_i, \hat{x}_i^k) - \frac{1}{k},$$

and hence

$$\begin{aligned} f_i(x_i^*, \hat{x}_i^*) &\geq \limsup_{k \to \infty} f_i(x_i^k, \hat{x}_i^k) \geq \lim_{k \to \infty} \left(\max_{u_i \in X_i} f_i(u_i, \hat{x}_i^k) - \frac{1}{k} \right) \\ &= \max_{u_i \in X_i} f_i(u_i, \hat{x}_i^*). \end{aligned}$$

Thus there is $x^* \in X$ such that for each $i \in N$,

$$f_i(x_i^*, \hat{x}_i^*) = \max_{u_i \in X_i} f_i(u_i, \hat{x}_i^*).$$

Theorem 4.1 is proved.

EXAMPLE. Let $X = [0,1] \times [0,1]$, take $f_1(x, y) = x^2 - \frac{1}{1+y}$, and $f_2(x, y) = y^2 - \frac{1}{1+x}$. Then for any $y \in [0,1]$, $f_1(\cdot, y)$ is Δ-quasiconcave; for any $x \in [0,1]$, $f_2(x, \cdot)$ is Δ-quasiconcave, and f_1, f_2 satisfy the other conditions of Theorem 4.1. The point $x^* = (1,1) \in X$ is a Nash equilibrium for the family of functions f_1 and f_2.

REFERENCES

1. K. Fan, A minimax inequality and its applications, *in* "Inequalities III" (O. Shisha, Ed.), pp. 103–113, Academic Press, San Diego, 1972.
2. C. D. Horvath and J. V. Llinares Ciscar, Maximal elements and fixed points for binary relations on topological ordered spaces, *J. Math. Econom.* **25** (1996), 291–306.
3. K. K. Tan and X. Z. Yuan, A minimax inequality with applications to existence of equilibrium points, *Bull. Austral. Math. Soc.* **47** (1993), 483–503.
4. E. Tarafdar, A fixed point theorem in H-space and related results, *Bull. Austral. Math. Soc.* **42** (1990), 133–140.
5. X. Wu and S. Shen, A further generalization of Yannelis–Prabhakar's continuous selection theorem and its applications, *J. Math. Anal. Appl.* **197** (1996), 61–74.
6. X. Z. Yuan, Knaster–Kuratowski–Mazurkiewicz theorem, Ky Fan minimax inequalities and fixed point theorems, *Nonlinear World* **2** (1995), 131–169.
7. J. Zhou and G. Tian, Transfer method for characterizing the existence of maximal elements of binary relations on compact sets, *SIAM J. Optim.* **2** (1992), 360–375.